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The maintenance of Reynolds stress in turbulent shear flow

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A mechanism is proposed for the manner in which the turbulent components support Reynolds stress in turbulent shear flow. This involves a generalization of Miles's mechanism in which each of the turbulent components interacts with the mean flow to produce an increment of Reynolds stress at the 'matched layer' of that particular component. The summation over all the turbulent components leads to an expression for the gradient of the Reynolds stress $\tau(z)$ in the turbulence

$$\frac{d\tau}{dz} = \mathscr{A}\Theta\overline{w^2}\frac{d^2U}{dz^2},$$

where $\mathscr{A}$ is a number, Θ the convected integral time scale of the w-velocity fluctuations and $U(z)$ the mean velocity profile. This is consistent with a number of experimental results, and measurements on the mixing layer of a jet indicate that $\mathscr{A} = 0{\cdot}24$ in this case. In other flows, it would be expected to be of the same order, though its precise value may vary somewhat from one to another.

1. Introduction

The mechanism that underlies the maintenance of Reynolds shear stress is at the heart of the dynamics of turbulent shear flow. In the flow through a circular pipe, for example, it provides an essential link between the axial pressure gradient and the wall drag. In addition, the kinetic energy of the fluctuating motion itself is maintained by the interaction of the Reynolds shear stress with the mean velocity gradient. The aim of this paper is to attempt to uncover the mechanisms that are involved in the generation of this Reynolds stress and to relate its characteristics to other measurable properties of the turbulent motion.

In the early days of this subject, it became customary to seek a relation between the local Reynolds stress and the local mean velocity field by the use of either an eddy viscosity or a mixing length defined in one way or another. But as more detailed experiments were performed by Townsend, Corrsin, Laufer, and others, it became increasingly evident that any such relation was erroneous in principle, since the Reynolds stress was found not to be a local property of the motion but one of the whole field of flow. It is probably not unfair to say that, at this point, the problem remained for a number of years. A very great step forward was taken by Townsend in 1956 in the publication of his monograph *The Structure of Turbulent Shear Flow*, where he presented an inductive account of the processes involved in the 'classical' turbulent shear flows. At about the same time Malkus (1956) offered a fresh view of certain aspects of the subject in which, by the use of a maximum energy dissipation principle, he was able to calculate at least the mean velocity profile in turbulent channel flow with extraordinary accuracy. The present analysis is much more mechanistic than Malkus's approach and our goals are more limited, being restricted simply to the consideration of the processes involved in the production of Reynolds stress by the turbulent motion.

In the last few years there have been a number of developments that have indicated the need for a further understanding of this question. One of these is the observation that the addition of small concentrations of long chain polymers to water can greatly reduce the pressure drop in flow through a pipe or the drag coefficient in a turbulent boundary layer. These substances have little effect on the overall viscosity of the fluid, but they do endow it with visco-elastic properties, and in some way these must influence the mechanism of the maintenance of Reynolds stress and so the pressure gradient in the flow. A similar effect has long been known in the turbulent flow of stratified fluids. When the mean density distribution is statically stable, the Reynolds stress is smaller than it would be in a homogeneous fluid with the same mean velocity field. It might be hoped that if the mechanism involved in the generation of Reynolds stress could be understood clearly, then some light would be shed on these observations also.

It is self-evident that the Reynolds stress must be generated by the interaction between the fluctuating motion and the mean velocity field. The simplest model that we might conceive concerns the interaction between a uniform shear and a single Fourier component of the turbulent velocity field. If the self-interaction of the superimposed sinusoidal disturbance is neglected, the problem is a linear one and a solution can be found without too much difficulty. The analyses of Pearson (1959), Deissler (1961) and Moffatt (1965) are of this kind and some interesting results emerge. One of these is that there appears to be *no* unique relation between the (local) mean velocity gradient and the Reynolds stress supported by the turbulence; it depends in detail on the structure of the turbulent components over the whole field. This analytical result is, of course, consistent with the failure of the simple 'eddy viscosity' idea mentioned earlier.

Such a model is, however, inherently deficient in its relevance to laboratory flows. It is known from experiment, Corrsin (1949), that the Reynolds stress is associated with the energy containing components of the turbulence, whose length scales are of the same order as the scale of the mean velocity variation. Over these distances the Reynolds stress varies, as does the mean velocity gradient; indeed it is the *variation* in Reynolds stress that enters the momentum equation and not the stress itself. Since the results from models with constant mean velocity gradient have been rather indecisive and disappointing, one is prompted to ask whether it might not be more fruitful to seek a possible association between the *variations* in Reynolds stress and those in mean velocity gradient.

In a quite different context, that of inviscid laminar air flow over water waves, such an association has been found. Miles (1957), in considering the generation of waves by wind, has discovered that a Reynolds stress is generated by the wave-induced air motion at the 'critical layer' where the wave speed equals the wind speed, and that the stress difference across this layer is proportional to the ratio of the local mean profile curvature to slope. This stress is then supported by the wave-induced undulations in the air flow below the 'critical layer', and continually transfers momentum to the waves. This discovery has been developed further by Miles himself (1962) and by Brooke Benjamin (1959, 1960); it has provided a cornerstone of our understanding of many problems involving the interaction of a mean and a small superimposed fluctuation motion. It is not unreasonable, then, to inquire whether this, or a similar process might be involved in the dynamics of shear flow turbulence itself. This was indeed suggested independently by Dr Benjamin as a post-script to a general lecture given to the Eleventh International Congress of Applied Mechanics at Munich in 1964. To be sure, the laminar analyses developed so far are still short of being able to cope with the problem of turbulence because of the co-existence in this case of many interacting 'disturbance' modes. Before the question can be answered, an essential first step is to determine whether this mechanism or one like it occurs also when a turbulent, not laminar, shear flow is subject to a small periodic perturbation. The lucid discussion given by Lighthill (1962) suggests very strongly that it does, an assertion that is reinforced by the rather different approach of the next section.

2. The matched layers

Let us consider the interaction between a turbulent shear flow and a superimposed, travelling, periodic velocity perturbation. The basic mean velocity field will, for the purposes of this section, be supposed to lie in the (x, y)-plane and to be a function of z alone; the orientation and velocity of the frame of reference being chosen so that the superimposed perturbation field is periodic in x, independent of y and time independent—the frame moves with the perturbation. The total velocity field can then be represented as

$$\mathbf{u} = \mathrm{U}(z) - \mathbf{c} + \mathfrak{U}(x, z) + \mathbf{u}'(x, y, z, t), \tag{2.1}$$

where $\mathrm{U}(z)$ is the mean velocity as observed in a frame of reference at rest and $\mathbf{c}$ the velocity with which the perturbation field moves through the fluid. The separation (2.1) can be achieved unambiguously by taking averages successively along lines parallel to the y-axis (an operation indicated by the symbol $\langle\,\rangle$) and over planes z = const. (indicated by an overbar). Thus

$$\langle \mathbf{u} \rangle = \mathrm{U}(z) - \mathbf{c} + \mathfrak{U}(x, z), \tag{2.2}$$

and
$$\bar{\mathbf{u}} = \overline{\langle \mathbf{u} \rangle} = \mathrm{U}(z) - \mathbf{c}. \tag{2.3}$$

Then $\mathfrak{U} = (\mathscr{U}, \mathscr{V}, \mathscr{W})$ represents the periodic perturbation field and $\mathbf{u}' = (u', v', w')$ the random velocity fluctuations of the turbulence.

The incompressibility condition is $\nabla . \mathbf{u} = 0$; the y-average of this equation yields

$$\frac{\partial}{\partial x}\{U(z)\cos\alpha - c + \mathscr{U}\} + \frac{\partial}{\partial z}\mathscr{W} = 0, \tag{2.4}$$

where α is the angle between the mean velocity vector and the x-axis. This is sufficient to ensure the existence of a stream function Ψ for the y-averaged motion such that

$$\left.\begin{aligned} U(z)\cos\alpha - c + \mathscr{U} &= \partial\Psi/\partial z, \\ \mathscr{W} &= -\partial\Psi/\partial x. \end{aligned}\right\} \tag{2.5}$$

Now, since the perturbation field $\mathscr{U}$ is periodic in x, the streamlines of the y-averaged motion can be represented as the real part of

$$\Psi = \int_{z_m}^{z} \{U(\xi)\cos\alpha - c\}\,d\xi + \psi(z)\,e^{ikx} = \text{const.}, \tag{2.6}$$

where the (arbitrary) lower limit of the integration is taken at the height z_m, where $U(z_m)\cos\alpha - c = 0$, or where the propagation speed of the disturbance just matches the component of the overall mean velocity field in the x-direction.

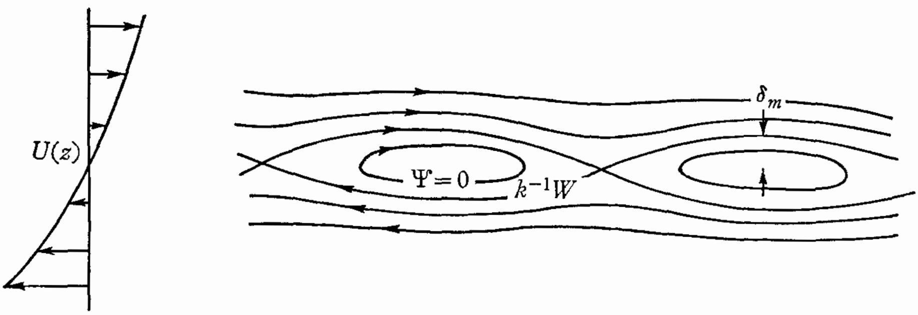

FIGURE 1. Streamlines of the mean flow in the neighbourhood of the matched layer. The distance δ_m characterizing the thickness of the layer is the maximum displacement of the streamline $\Psi = k^{-1}W(z_m)$.

In regions distant from z_m, the streamlines of the y-averaged flow are merely smooth undulations on a uniform stream. If

$$\mathrm{Re}\,\{\psi(z)\,e^{ikx}\} = -k^{-1}W(z)\cos[kx + \epsilon(z)]$$

(so that $W(z)$ is the amplitude of the $\mathscr{W}$-perturbation), it can be shown simply that the displacement of a mean streamline about its average height z_1 is

$$\delta = z - z_1 = \frac{W(z_1)\cos[kx + \epsilon(z_1)]}{k[U(z_1)\cos\alpha - c]}. \tag{2.7}$$

Near z_m, however, where the integrand in (2.6) vanishes, Ψ can be approximated by

$$\Psi = \tfrac{1}{2}(z - z_m)^2\,U'(z_m)\cos\alpha - k^{-1}W(z_m)\cos[kx + \epsilon(z_m)]. \tag{2.8}$$

The streamlines $\Psi = \text{const.} \leqslant k^{-1}W(z_m)$ now represent closed loops centred on the height z_m as illustrated in figure 1. The existence of these loops is simply a kinematical consequence of the non-vanishing of $\mathscr{W}$ at the height z_m.

It is important to remember that these mean streamlines do not coincide at all with the particle paths; the flow is turbulent, and superimposed on the mean streaming are the random velocity fluctuations whose amplitudes may well be large compared with $\mathscr{U}$ or $\mathscr{W}$. The position z_m, where the mean speed is just equal to the wave speed of the perturbation, marks, in the context of laminar instability theory, the position of the 'critical layer'. In that situation, when the Reynolds number is large, the critical layer is a thin region where the vorticity perturbation is very large; it plays a central role in the development of an instability. In the present problem, the region near z_m will be found to be equally important, but it may not be particularly thin, nor may the vorticity of the wave-induced perturbation be especially large. To avoid confusion, and to reserve for the term 'critical layer' its well established meaning in stability theory, this part of the flow will be called the *matched layer*.

The thickness δ_m of the matched layer is conveniently represented by the maximum displacement of the mean streamline $\Psi = W(z_m)/k$, so that from (2.8),

$$\delta_m = \left\{\frac{4W(z_m)}{kU'(z_m)\cos\alpha}\right\}^{\frac{1}{2}}. \tag{2.9}$$

There is a simple and important connexion between the Reynolds stress supported by the mean perturbation field $\mathfrak{U}$ and its vorticity

$$\Omega = \frac{\partial \mathscr{U}}{\partial z} - \frac{\partial \mathscr{W}}{\partial x}.$$

For

$$\Omega\mathscr{W} = \left(\frac{\partial \mathscr{U}}{\partial z} - \frac{\partial \mathscr{W}}{\partial x}\right)\mathscr{W}$$

$$= \frac{\partial}{\partial z}\mathscr{U}\mathscr{W} + \frac{\partial}{\partial x}(\tfrac{1}{2}\mathscr{U}^2 - \tfrac{1}{2}\mathscr{W}^2),$$

since $\partial\mathscr{W}/\partial z = -\partial\mathscr{U}/\partial x$. The overall (or x) average of this expression yields

$$\frac{\partial}{\partial z}\overline{\mathscr{U}\mathscr{W}} = \overline{\Omega\mathscr{W}} \tag{2.10}$$

and if the perturbation field vanishes at infinity, say,

$$\tau_p = -\rho\overline{\mathscr{U}\mathscr{W}} = \rho\int_z^\infty \overline{\Omega\mathscr{W}}\,dz. \tag{2.11}$$

This integral provides a means for estimating the Reynolds stress supported by the periodic perturbation. The basic mean flow has a vorticity distribution $U'(z)\cos\alpha$ normal to the plane of the perturbation, and the periodic disturbance represents a small undulation in this. The *variation* Ω in mean vorticity $\langle\omega\rangle$ at a fixed height is thus proportional to the magnitude of the undulations and to the mean vorticity gradient

$$|\Omega| \propto |-U''(z)\,\delta\cos\alpha|, \tag{2.12}$$

provided $\delta \ll |U''(z)/U'''(z)|$. Away from the matched layer, δ is given by (2.7), so that the covariance $\overline{\Omega\mathscr{W}}$ takes the form

$$\overline{\Omega\mathscr{W}} = A\frac{-U''(z)\,\overline{\mathscr{W}^2}(z)\cos\alpha}{k|U(z)\cos\alpha - c|}, \tag{2.13}$$

where the dimensionless number A is proportional to the correlation coefficient between the variations Ω in vorticity and $\mathscr{W}$ in cross-stream velocity. Now, in laminar, inviscid flow, the vorticity is conserved along each streamline, so that Ω and $\mathscr{W}$ are in exact quadrature and their correlation coefficient is zero. Consequently, regions of the flow away from the matched layer make no contribution at all to the integral (2.11). In *turbulent* flow, on the other hand, the motion is highly diffusive and vorticity is not, in general, conserved. Nevertheless, in this region, the whole flow undulates slightly and the distance between neighbouring mean streamlines changes little so that the mean vorticity might still be expected to be very nearly constant along the mean streamlines. As a result, the correlation coefficient A would be expected to be small in this case also. In view of the contributions to the integral (2.1) from the region of the matched layer, which, as is shown below, are certainly significant, it is proposed in this context to assume that A is negligibly small.

In the matched layer, on the other hand, the mean streamlines given by (2.8) represent closed loops and the variations in vorticity Ω and $\mathscr{W}$ can be expected to be highly correlated. The non-vanishing of this correlation is, in fact, the basic dynamical hypothesis of this paper. The thickness of the layer, δ_m, is given by (2.9) and the variations in mean vorticity are proportional to $-U''(z_m)\,\delta_m\cos\alpha$. Thus

$$(\overline{\Omega\mathscr{W}})_m \propto -U''(z_m)\,W(z_m)\,\delta_m\cos\alpha.$$

The contribution to the integral (2.11) from the matched layer associated with this perturbation field is therefore

$$\Delta\tau_p \simeq \rho(\overline{\Omega\mathscr{W}})_m\,\delta_m \propto -\rho U''(z_m)\,W(z_m)\,\delta_m^2\cos\alpha,$$

or, from (2.9)

$$\Delta\tau_p = A_m\rho\left\{\frac{-U''}{kU'}\overline{\mathscr{W}^2}\right\}_{z_m}, \tag{2.14}$$

where A_m is a numerical constant. Since $(\overline{\Omega\mathscr{W}})_m$ is positive when the basic mean vorticity $U'(z)$ decreases with z, A_m is positive. This result was derived originally by Miles (1957) from an inviscid, laminar model. This discussion indicates that it remains true in turbulent flow although the numerical value of the constant A_m (which is π in Miles's theory) is undetermined. Near a solid wall, however, this expression would be expected to become inaccurate for two reasons. In the first place, the mean velocity gradient in (2.8) changes rapidly across the matched layer—the streamline pattern loses its symmetry. Secondly, the mean *vorticity* gradient also changes rapidly within this region, and the further terms in the Taylor series expansion (of which (2.12) is only the first term) become dominant.

3. The Reynolds stress gradients in turbulence

Miles's formula (2.14) was derived originally in the context of the air flow over surface waves. But in as far as the processes involved in the generation of Reynolds stress in the matched layers are concerned, the particular association of the perturbation with surface waves is not essential; the *same* mechanism will be involved for each of the component fluctuations of the turbulence itself. The turbulent motion can be regarded in a Fourier decomposition as the superposition of a large number of small travelling perturbations, spatially periodic in the (x, y)-plane, each of which will interact with the mean flow in the manner described above.

The turbulent velocity field in the shear flow will be represented as

$$u_i(\mathbf{x}, t) = \int_{\mathbf{k}}\int_n dB_i(\mathbf{k}, n, z)\, e^{i(\mathbf{k}\cdot\mathbf{x}+nt)}, \tag{3.1}$$

where the integration is over the horizontal (say) wave-number plane and over all frequencies n. For the sake of simplicity, the turbulence is supposed statistically homogeneous in horizontal planes, though not in the vertical. The Fourier–Stieltjes transform dB_i is also, of course, a function of the vertical co-ordinate z. It is important to notice that the wave-number $\mathbf{k}$ and frequency n are both real, so that for each component $dB(\mathbf{k}, n, z)$ there corresponds a *real* propagation velocity $-n\mathbf{k}/k^2$ in some horizontal direction. This is not to say that all components propagate (or are convected) at the same speed; for a given $\mathbf{k}$ there will be found in the turbulence a range of frequencies n over which there are significant amplitudes $|\alpha B(\mathbf{k}, n, z)|$ and so a range of convection velocities. But for any given component in the decomposition (3.1), with $\mathbf{k}$ and n both specified, the convection velocity is unique. Provided this is equal to the component in the direction of $\mathbf{k}$ of the mean velocity at some height z in the flow, there exists a matched layer for this component with its associated contribution to the Reynolds stress.

Consider, then, the component with some given wave-number $\mathbf{k}$ having its matched layer at some given point z_m, that is, having the particular frequency

$$n_m = -\mathbf{k}\,.\,\mathbf{U}(z_m). \tag{3.2}$$

The contribution to the mean square vertical velocity fluctuation ($\overline{\mathscr{W}^2}$ in equation (2.14)) from components in the range (dk, dn) about $\mathbf{k}$, n_m is given by

$$\Psi_{33}(\mathbf{k}, n_m, z)\, d\mathbf{k}\, dn,$$

where

$$\begin{aligned}\Psi_{33}(\mathbf{k}, n_m, z) &= (2\pi)^{-3}\iint \overline{u_3(x, y, z, t_0)\, u_3(x+r_1, y+r_2, z, t_0+t)}\, e^{-i(\mathbf{k}\cdot\mathbf{r}+n_m t)}\, d\mathbf{r}\, dt \\ &= \frac{\overline{dB_3(\mathbf{k}, n_m, z)\, dB_3^*(\mathbf{k}, n_m, z)}}{dk_1\, dk_2\, dn}.\end{aligned} \tag{3.3}$$

From (2.14), the increment of Reynolds stress supported by this small range of components is

$$A_m\, \rho \frac{-U''(z_m)}{kU'(z_m)} \Psi_{33}(\mathbf{k}, n_m, z)\, d\mathbf{k}\, dn \tag{3.4}$$

in the direction of the wave-number $\mathbf{k}$, or this times $\cos\alpha$ in the direction of the mean stream $\mathbf{U}$. For the components of a given wave-number, the range of frequencies dn about n_m corresponds to a range dz in the position of the matched layer, for, from (3.2),

$$dn = -kU'(z_m)\cos\alpha\, dz.$$

Thus

$$d\tau_k = A_m \rho U''(z_m)\cos^2\alpha\, \Psi_{33}(\mathbf{k}, n_m, z)\, d\mathbf{k}\, dz,$$

and the contribution from *all* the wave-numbers of the turbulence with their matched layers at z_m is

$$d\tau = A_m \rho U''(z_m)\,dz \int \cos^2\alpha \Psi_{33}(\mathbf{k}, n_m, z)\,d\mathbf{k}, \tag{3.5}$$

the integration being over all wave-numbers and correspondingly over frequencies on the plane $n_m = -\mathbf{k}.\mathbf{U}(z_m)$ in wave-number, frequency space.

This expression gives the increment of Reynolds stress τ over a small range dz about *any* level z_m, so that it can be written

$$\frac{d\tau}{dz} = \mu_e \frac{d^2U}{dz^2}, \tag{3.6}$$

where μ_e, the apparent 'eddy viscosity'† is

$$\mu_e(z) = A_m \rho \int \cos^2\alpha \Psi_{33}(\mathbf{k}, n_m, z)\,d\mathbf{k}. \tag{3.7}$$

Before the implications of this rather surprising result are discussed, it will be shown how (3.7) can be interpreted simply in terms of measurable physical properties of the turbulence.

From its definition (3.3), $\Psi_{33}(\mathbf{k}, n, z)$ is the Fourier transform of the covariance between the vertical velocity fluctuations $u_3 \equiv w$ at the level z at points with *horizontal* separation $\mathbf{r}$ and time delay t. The inverse relation is

$$\overline{w(\mathbf{x}, t_0)\,w(\mathbf{x}+\mathbf{r}, t_0+t)} = \iint \Psi_{33}(\mathbf{k}, n, z)\,e^{i(\mathbf{k}.\mathbf{r}+nt)}\,d\mathbf{k}\,dn.$$

If, however, the covariance is measured in a frame of reference moving with the local mean velocity $\mathbf{U}(z)$, then in this frame the spatial separation between the two points is

$$\mathbf{r}' = \mathbf{r} - \mathbf{U}(z)\,t,$$

and $$\overline{w(\mathbf{x}, t_0)\,w(\mathbf{x}+\mathbf{r}'+\mathbf{U}(z)\,t, t_0+t)} = \iint \Psi_{33}(\mathbf{k}, n, z)\,e^{i(\mathbf{k}.\mathbf{r}')+i(\mathbf{k}.\mathbf{U}+n)t}\,d\mathbf{k}\,dn.$$

Finally, if $\mathbf{r}'$ is taken as zero (that is, if the covariance is measured as a function of time at a point moving with the mean velocity of the fluid at the level z), this expression reduces to

$$\begin{aligned}\overline{w(\mathbf{x}, t_0)\,w(\mathbf{x}+\mathbf{U}(z)\,t, t_0+t)} &= \overline{ww'}, \quad \text{say},\\ &= \iint \Psi_{33}(\mathbf{k}, n, z)\,e^{i(\mathbf{k}.\mathbf{U}+n)t}\,d\mathbf{k}\,dn.\end{aligned}$$

Thus $$\begin{aligned}\int_{-\infty}^{\infty} \overline{ww'}\,dt &= \int_{-\infty}^{\infty} \left\{\iint \Psi_{33}(\mathbf{k}, n, z)\,e^{i(\mathbf{k}.\mathbf{U}+n)t}\,d\mathbf{k}\,dn\right\} dt\\ &= \iint \Psi_{33}(\mathbf{k}, n, z)\,\delta(\mathbf{k}.\mathbf{U}+n)\,d\mathbf{k}\,dn\\ &= \int_{\mathbf{k}} \Psi_{33}(\mathbf{k}, -\mathbf{k}.\mathbf{U}, z)\,d\mathbf{k},\end{aligned} \tag{3.8}$$

† Note that this is not the usual definition.

from the theory of generalized functions (see, e.g. Lighthill 1958), where δ represents the Dirac delta function. The integral on the left can be expressed in terms of the integral time scale Θ of the w-velocity fluctuations in a frame of reference moving with the local mean velocity:

$$\int_0^\infty \overline{ww'}\,dt = \overline{w^2}\Theta, \tag{3.9}$$

so that

$$\int_{\mathbf{k}} \Psi_{33}(\mathbf{k}, -\mathbf{k}.\mathbf{U}, z)\,d\mathbf{k} = 2\overline{w^2}\Theta. \tag{3.10}$$

This integral is identical with that in (3.7) except for the factor $\cos^2\alpha$ that suppresses contributions from the wave-numbers that are nearly normal to the plane of shear. The ratio of the one integral to the other depends on the directional distribution of Ψ_{33} in the $\mathbf{k}$-plane; if, for example, the w-velocity fluctuations were statistically isotropic in the planes z = const., then the integral in (3.7) would be just one half of (3.10). In fact, these fluctuations are far from isotropic, but in any event, the ratio is a pure number that may vary to some extent from one flow to another. Consequently (3.7) can be written as

$$\mu_e(z) = \mathscr{A}\rho\overline{w^2}\Theta, \tag{3.11}$$

where $\mathscr{A}$ is a dimensionless constant. Because of the anisotropy of shear flow turbulence, with eddies elongated in the flow direction, the spectral density of wave-numbers near $\alpha = \pm\frac{1}{2}\pi$ is large, and consequently $\mathscr{A}$ must be expected to be significantly smaller than A_m.

These results (3.6) and (3.11) have diverse implications. An important principle that they display is that the Reynolds stress is *not* a local property of the fluid motion, as has long been known experimentally. They do assert, however, that in a sense the stress *gradient* is a local property to this approximation; it involves only the properties of the motion at the height z but it does require specification of Θ, which is dependent on the time history of the turbulence. Also, they show quite clearly the importance of the energy-containing eddies in supporting the Reynolds stress—the 'eddy viscosity' is in fact proportional to the kinetic energy density of the vertical velocity fluctuations. Finally, the appearance of the convected integral time scale Θ in (3.11) shows that the longer the w-fluctuations remain coherent, the greater are their contributions to the stress gradient.

Similar expressions can be derived for flow in a pipe, which is statistically homogeneous in the axial direction and which possesses axial symmetry. The steps of the previous two sections must be repeated with due regard for the configuration. The response must be found of the turbulent motion to a perturbation, in which in general the lines of constant phase are helices. The relation analogous to (2.10) is found to be

$$\overline{\Omega\mathscr{W}} = -\frac{1}{r}\frac{d}{dr}(r\overline{\mathscr{U}\mathscr{W}}),$$

where Ω here represents the variation in the vorticity component directed along the lines of constant phase, $\mathscr{W}$ represents the radial component of the perturbed velocity and $\mathscr{U}$ the component orthogonal to these two directions. The variations in this vorticity component are proportional to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)\delta\cos\alpha$$

(cf. equation (2.12)), where $U(r)$ is the mean axial velocity gradient. The details will not be given here since the ideas are in essence the same, though the analytical representations are a little less simple. The end result is that

$$\frac{d}{dr}(r\tau_{rx}) = \mu_e\frac{d}{dr}\left(r\frac{dU}{dr}\right), \tag{3.12}$$

where $\tau_{rx} = -\rho\overline{uw}$

is the Reynolds stress in the axial direction across an element of area normal to the radius and μ_e has the same form as (3.11).

4. Some applications

The results presented in the previous section may be useful in three ways. Besides displaying the way in which an 'eddy viscosity' emerges naturally from this mechanism involved in the generation of Reynolds stress, they allow some immediate predictions and comparisons with experiments and also provide a basis for further hypothesis and approximation.

One immediate inference is that, provided Θ does not vanish, the Reynolds stress in the interior of a turbulent shear flow with a mean velocity distribution $U(z)$ has extrema when and only when the profile curvature vanishes, regardless of the variation of μ_e with z. There are a number of flows in which this prediction can be compared with experiment. Probably the most striking is turbulent plane Couette flow, in which the shear stress is independent of z, and so from (3.6), the mean velocity profile is linear in the central region of the flow. This is a rather difficult motion to achieve experimentally, but some mean profile measurements have been made by Robertson (1959). These are summarized in figure 2, and although their precision is not high, it it evident that outside the wall regions, the mean velocity gradient in all cases is nearly independent of z. Another flow in which the mean velocity field is very nearly a function of z alone is the turbulent wake of a cylinder and Townsend (1956) has summarized a number of experimental studies. From these, it is found that τ is a maximum when $z = 0{\cdot}20[(x-x_0)\,d]^{\frac{1}{2}}$, where $(x-x_0)$ is the distance downstream from the virtual origin and d is the cylinder diameter. The point of inflexion in the mean velocity profile occurs at $z = 0{\cdot}21\,[(x-x_0)\,d]^{\frac{1}{2}}$. Again, in a plane jet, Bradbury's (1965) measurements show that a broad maximum in the shear stress occurs in the neighbourhood of $z = 0{\cdot}75\delta$ where δ is the local jet thickness. The point of inflexion of the mean profile is less well defined, but lies between $0{\cdot}6\delta$ and δ. In boundary layer, pipe and channel flows, the Reynolds stress gradient vanishes nowhere in the interior of the fluid, and the mean velocity gradient is monotonic.

These points of agreement are encouraging, but there are some more quantitative comparisons that can also be made. If, for symmetry or other reasons, the

Reynolds stress vanishes at some point z_0 where the mean velocity gradient is zero (for example, at the centre line of a channel or of a cylinder wake), then (3.6) can be integrated by parts to give

$$\tau(z) = \mu_e(z)\frac{dU}{dz} - \int_{z_0}^{z} \frac{d\mu_e(z)}{dz}\frac{dU}{dz}dz. \tag{4.1}$$

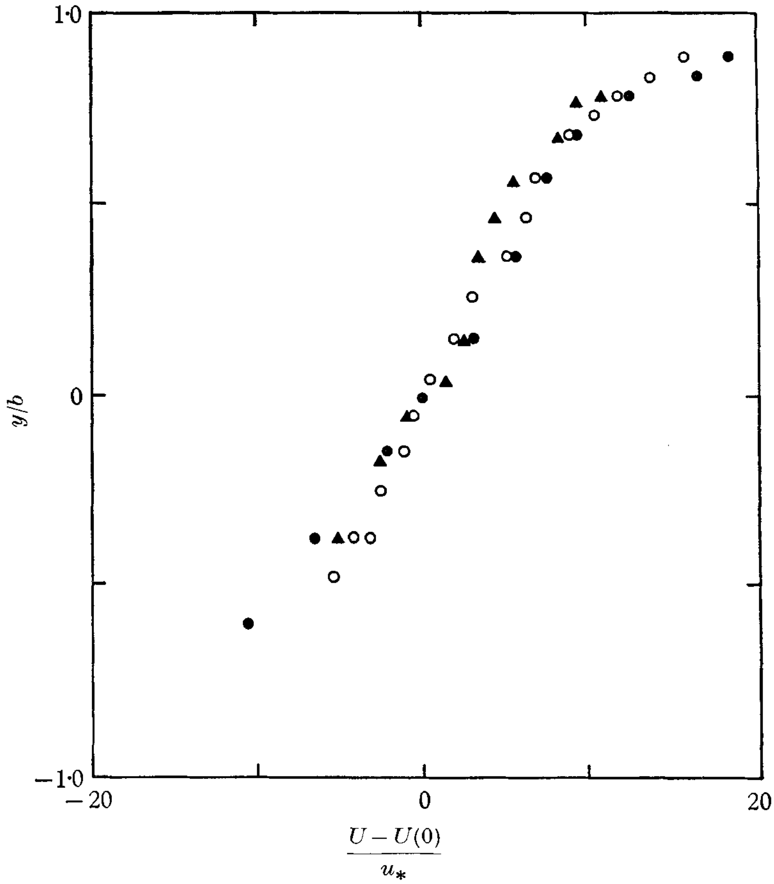

FIGURE 2. Mean velocity distribution in plane Couette flow, with distance $2b$ between plates. The measurements are by Robertson (1959); ○, at Reynolds number $R = U(0)\,b/\nu = 1{\cdot}5\times10^4$; ▲, at $R = 1{\cdot}1\times10^4$ and ●, at $R = 5{\cdot}8\times10^3$.

If μ_e is independent of z over the interval (or over the part of it in which the mean velocity varies) then the last term vanishes and (4.1) reduces to the expression conventionally used to define the 'eddy viscosity'. It is well known (see, for example, Townsend 1956) that the assumption of constant eddy viscosity in most free turbulent flows (with due allowance for the intermittency near the outside edge) leads to velocity profiles in remarkably good agreement with experiment, and although this might not have been *deduced* from these expressions, it is certainly consistent with them. Of greater interest, however, is the magnitude of the 'eddy viscosity' found experimentally and the light that this sheds on the magnitude of the dimensionless number $\mathscr{A}$.

Because of its importance in questions involving the aerodynamic generation of sound, the shear layer of a jet is one flow in which careful measurements have been made of the mean velocity profile, the turbulent intensities and the convected integral time scales. The experiments of Davies, Fisher & Barratt (1963) are most useful. From their observations, it appears that in the central region of the shear layer, the second term of (4.1) with z_0 at $-\infty$ is numerically small compared with the first, so that in this case,

$$\tau(z) \simeq \mathscr{A}\overline{w^2}\Theta\, dU/dz. \tag{4.2}$$

They measured the integral time scale for longitudinal fluctuations and found that $\Theta \simeq 3{\cdot}2(dU/dz)^{-1}$ at different points across the flow. If the corresponding quantity for the lateral fluctuations is not appreciably different, then (4.2) becomes simply

$$\tau(z) \simeq 3{\cdot}2\mathscr{A}\overline{w^2}. \tag{4.3}$$

The figures of table 1 are taken from Townsend's (1956) book, the notation being modified slightly to conform to our present usage. The remarkable constancy of the ratio $\tau/3{\cdot}2\overline{w^2}$ in this particular flow in the region $-0{\cdot}25x \leqslant z \leqslant 0{\cdot}75x$

z/x	$\tau/\overline{u^2}$ (Townsend)	$\overline{u^2}/U_1^2$ (Townsend)	τ/U_1^2	$\overline{w^2}/U_1^2$ (Townsend)	$\dfrac{\tau}{3{\cdot}2\overline{w^2}} = \mathscr{A}$
−0·50	0·41	0·0080	0·00328	0·0060	0·170
−0·25	0·41	0·0165	0·00675	0·0085	0·248
0·00	0·39	0·0210	0·00820	0·0100	0·256
0·25	0·30	0·0215	0·00645	0·0090	0·224
0·50	0·23	0·0175	0·00400	0·0050	0·250
0·75	0·15	0·0125	0·00187	0·0025	0·234
1·00	0·10	0·0065	0·00065	0·0015	0·134

TABLE 1

appears to support the theoretical ideas described here, the scatter about the mean value $\mathscr{A} = 0{\cdot}24$ being of the order 5 %. Outside this region, the flow is more highly intermittent and the mean turbulent intensity is small so that the ratios are less accurate. Moreover, the second term of (4.1) is no longer negligible here so that the approximation (4.2) is inadequate. It need be no surprise then that the values of the ratio $\tau/3{\cdot}2\overline{w^2}$ at $z = -0{\cdot}50x$ and $z = +1{\cdot}00x$ are significantly different from those in the central region.

This value, $\mathscr{A} = 0{\cdot}24$, is then characteristic of the turbulent mixing layer of a jet. The corresponding number in other flows would be expected to be of the same order, though not necessarily identical because of a possibly different degree of anisotropy in the (x, y)-plane of the energy-containing eddies. It is unfortunate that measurements of the integral time scale Θ in other flows do not seem to have been reported yet—it is certain that Davies, Fisher & Barratt's simple relation between Θ and dU/dz cannot be true universally. The closest approach is found in the results of Favre, Gaviglio & Dumas (1957, 1958) on the turbulent boundary layer, but their space-time correlation measurements extend only to correlations of about 0·4, which is still too large to provide a sound estimate of Θ.

There are also the surface pressure fluctuation measurements by Willmarth & Wooldridge (1962) and Corcos (1964) in which space-time correlations were made, but it would be hazardous to try to estimate from these the integral time scale of the w-velocity fluctuations at different points in the flow itself. Though one would expect $\mathscr{A}$ to be of the order 0·24 in other flows, also, it appears that confirmation (or denial!) of this expectation must await further measurement.

Finally, it might be indicated how these results could be used as a basis for further hypothesis. In the turbulent flow of a stratified fluid in which the mean density $\bar{\rho}$ is a function of z, the large scale 'packages' of fluid will tend to oscillate about their mean level with the Brunt–Väisälä frequency

$$N = \left\{-\frac{g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}\right\}^{\frac{1}{2}},$$

where g is the gravitational acceleration. The time correlation between the vertical velocity fluctuations in a frame of reference moving with the local mean velocity might perhaps be supposed to be of the form $f(t)\cos Nt$, where

$$\int_0^\infty f(t)\,dt = \Theta$$

the integral time scale is an unstratified flow. If N is of the order Θ^{-1} or greater, the integral time scale in the stratified fluid

$$\Theta(N) = \int_0^\infty f(t)\cos Nt\,dt$$

is considerably less than in the equivalent flow of a homogeneous fluid, so that the effective 'eddy viscosity' is immediately reduced. This in turn reduces the energy flux from the mean flow and the turbulent intensity, further reducing μ_e. But the proper formulation of these ideas is still some distance away, and they will not be pursued here.

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